Collapse and revival oscillations as a probe for the tunneling amplitude in an ultra-cold Bose gas

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27 July 2011

Collapse and revival of the matter wave field of a Bose-Einstein condensate

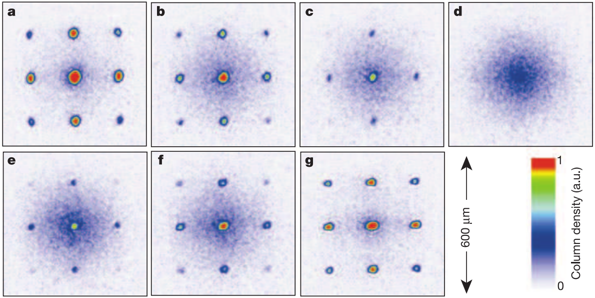
M. Greiner, O. Mandel, T. W. Hänsch, and I. Bloch, Nature **419** (2002)

scenario

initital state = BEC, approx. by a single-site coherent state: $|\alpha_0\rangle = e^{|\alpha|} \sum_n \frac{\alpha^n}{n!} |n\rangle$

hamiltonian after quench: $\hat{H}(t \geq 0) = \frac{1}{2} U \hat{n}(\hat{n} - 1)$

$\Rightarrow$ periodic time evolution with frequency $\omega = U$: $|\alpha(t)\rangle = e^{|\alpha|} \sum_n e^{-i\frac{1}{2}Un(n-1)t} \frac{\alpha^n}{n!} |n\rangle$



$\left\langle a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}} \right\rangle$: matter wave interference pattern in the k_x-k_y-plane for different times t

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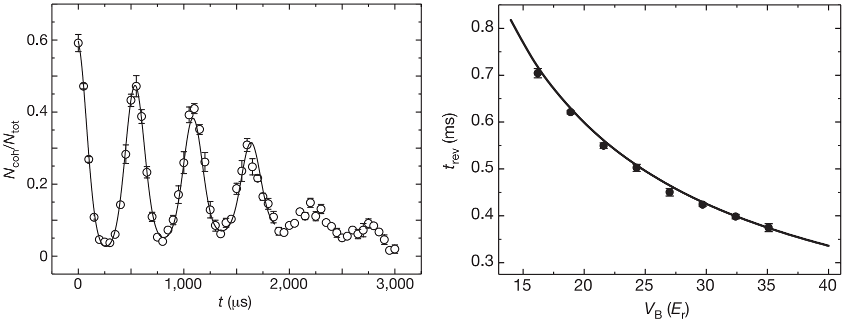


$N_{coh}/N_{tot} \propto \left\langle a^\dagger_{k=0} a_{k=0} \right\rangle$

frequency $\omega = U \Rightarrow t_{rev} = 2\pi/U$

Collapse and revival oscillations as a probe for measuring multi-body interaction energies

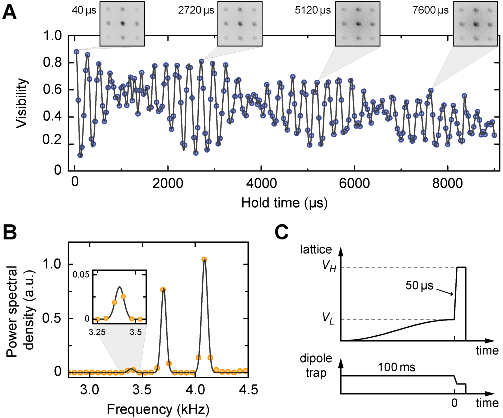
multi-body interactions show rescaling of two-body interaction constant: $U \equiv U_2 \to U_3, U_4 \ldots$

theoretical proposition:

P. R. Johnson *et al.*, N. J. Phys. **11** (2009)

experimental realization:

S. Will *et al.*, Nature **465** (2010) $\to$ Figure



A: "visibility" $\propto \left\langle a^\dagger_{k=0} a_{k=0} \right\rangle$

B: fourier analysis of interference pattern

C: time-scales and lattice depth of experimental realization for quench

Matter of fact

Former investigations of collapse and revivals only for systems in the atomic limit or via a solely meanfield approach.

Now: extensive study of the phenomenon to **extract the influence of the hopping amplitude** using full many body states by application of

- exact techniques to estimate the predicitve power of
- a Gutzwiller mean-field approach for systems with a large Hilbert space

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Outline

- Exact approaches to hard-core bosons in non-equilibrium
- Gutzwiller mean-field approach vs. exact results
- Results for experimentally relevant systems

Exact approaches to hard-core bosons in non-equilibrium

Why study hard-core bosons on a superlattice?

Bosons on an optical lattice are well described by the Bose-Hubbard model

$$\hat{H}_{\text{SCB}} = -J \sum_{\langle ij \rangle} (\hat{b}_i^\dagger \hat{b}_j + \text{H. c.}) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) + V \sum_i r_i^2 \hat{n}_i$$

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What are hard-core bosons?

$$\hat{H}_{\mathsf{HCB}} = -J \sum_{\langle ij \rangle} \left(\hat{c}_i^\dagger \hat{c}_j + \mathrm{H.c.}\right) + V \sum_i r_i^2 \hat{n}_i$$

where $[\hat{c}_i, \hat{c}_j^\dagger] = \delta_{ij}$, $[\hat{c}_i, \hat{c}_j] = 0$ and $\hat{c}_i^\dagger \hat{c}_i^\dagger = 0$

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So, why study hard-core bosons on a superlattice?

$$\hat{H}_{\text{HCB}} = -J \sum_{\langle ij \rangle} \left(\hat{c}_i^\dagger \hat{c}_j + \text{H.c.} \right) + A \sum_i (-1)^i \hat{n}_i + V \sum_i r_i^2 \hat{n}_i$$

Because HCBs on a super-lattice show similar physical phenomena as compared to SCBs but numerically exact solutions are available.

Hard-core bosons on a superlattice

Two bands similiar to Hubbard bands

- diagonalization of $\hat{H}_{\text{HCB}}$ by means of a fourier transform yields

$$\epsilon_{\pm}(k) = \pm\sqrt{4J^2\cos^2(ka) + A^2}$$

- superlattice A opens gap for HCBs as interaction U does for SCBs

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Consequences

- equilibrium: similar phase diagram to that of the Bose Hubbard model
  - I. Hen and M. Rigol, Phys. Rev. B **80** (2009)
  - I. Hen, M. Iskin, and M. Rigol, Phys. Rev. B **81** (2010)
- non-equilibrium: collapse and revival oscillations, A plays role of U
  - M. Rigol, A. Muramatsu, and M. Olshanii, Phys. Rev. A **74** (2006)

Hard-core bosons in one dimension

Map on free fermions by Jordan-Wigner transformation

$$c_j^\dagger = a_j^\dagger \prod_{\beta=1}^{j-1} e^{-i\pi a_\beta^\dagger a_\beta}$$

- calculation of properities of non-interacting particles through one-particle representation of hamilton operator: Hilbert space dimension = L
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- non-equilibrium properties for system sizes with $L \sim 500$
- investigation of inhomogeneous (trapped) systems possible

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Hard-core bosons in two dimensions

Exact Diagonlization

- system size $L = 4 \times 4 = 16$
- small but meaningfull for periodic systems
- not meaningfull for trapped case

Results for hard-core bosons in one and two dimensions

observable

$$n_{k=0} = \frac{1}{L} \sum_{ij} \langle \hat{b}_i^\dagger \hat{b}_j \rangle$$

revival time

$$\Delta t_{\text{rev}} = t_{\text{rev}}^{\text{atom}} - t_{\text{rev}}$$

$$t_{\text{rev}}^{\text{atom}} = \pi/A, \; A \equiv 1$$

revival amplitude

$$\Delta n_{k=0}^{\text{rev}} = n_{k=0\,\text{rev}}^{\text{atom}} - n_{k=0}^{\text{rev}}$$

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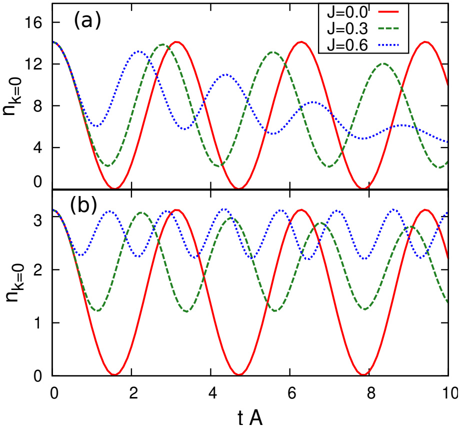

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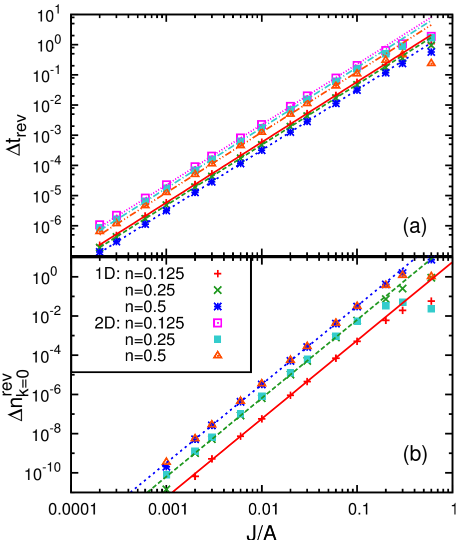
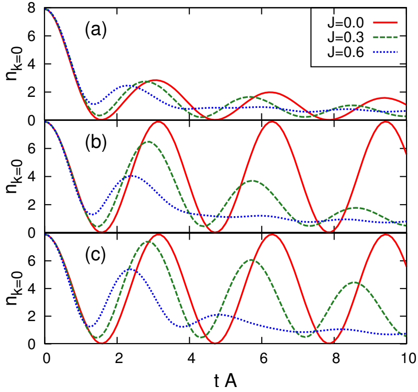


Results for hard-core bosons for a trapped system in 1D

$\tilde{\rho} = N[V/(dJ)]^{\frac{d}{2}}$ compare

$$\hat{H}_{\text{HCB}} = -J \sum_{\langle ij \rangle} \left(\hat{b}_i^\dagger \hat{b}_j + \text{H.c.}\right) + V \sum_i r_i^2 \hat{n}_i$$



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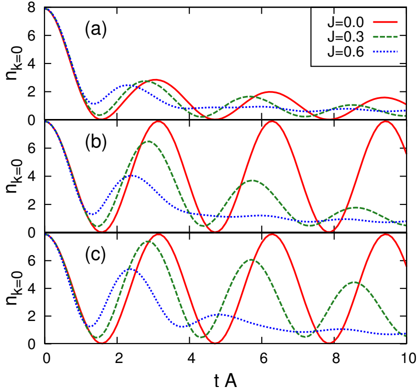

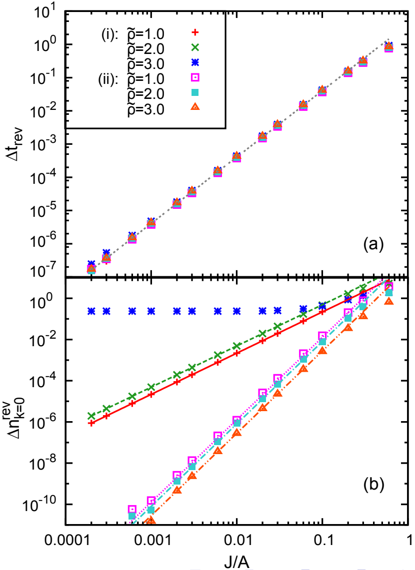

Gutzwiller mean-field approach vs. exact results

Gutzwiller mean-field approach

Back to the Bose-Hubbard model

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Gutzwiller type product state

$$|\Psi_{\mathrm{MF}}\rangle = \prod_{i=1}^{L} \Big(\sum_{n=1}^{n_{\mathsf{c}}} \alpha_{in} \frac{(b_i^\dagger)^n}{n!} \Big) |0\rangle = \prod_{i=1}^{L} \Big(\sum_{n=1}^{n_{\mathsf{c}}} \alpha_{in} |n\rangle_i \Big)$$

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Variational principle

$$\delta \, \langle \Psi_{\mathrm{MF}} | \hat{H}_{\mathsf{SCB}} - \mu \hat{N} | \Psi_{\mathrm{MF}} \rangle = 0$$

Gutzwiller mean-field approach for non-equilibrium

D. Jaksch, V. Venturi, J. I. Cirac, C. J. Williams, and P. Zoller, Phys. Rev. Lett. **89** (2002)

Time-dependent variational principle

$$\delta \langle \Psi_{\text{MF}} | i\partial_t - \hat{H}_{\text{SCB}} + \mu \hat{N} | \Psi_{\text{MF}} \rangle = 0$$

yields set of $L \times n_{\text{c}}$ differential equations

$$i\dot{\alpha}_{in} = -J \sum_{\langle j \rangle_i} \left(\sqrt{n+1}\,\alpha_{i(n+1)} \Phi_j^* + \sqrt{n}\,\alpha_{i(n-1)} \Phi_j \right) + \alpha_{in}\, n \left[\frac{U}{2}(n-1) + V r_i^2 - \mu \right]$$

where $\Phi_j = \langle a_j \rangle = \sum_{n=1}^{n_{\text{c}}} \sqrt{n}\,\alpha^*_{j(n-1)} \alpha_{jn}$

- numerically solved with forth-order Runge-Kutta method

Analytical solution for HCBs in the mean-field approach

due to mapping on spin-states

$$|\Psi_{\mathsf{MF}}^{\mathsf{HCB}}\rangle = \prod_{i=1}^{L} \mathrm{e}^{i\chi_i} \left(\sin\frac{\theta_i}{2} + \cos\frac{\theta_i}{2} \mathrm{e}^{i\phi_i} b_i^\dagger \right) |0\rangle$$

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and translational invariance (two-site problem) $\rightarrow$ massive simplification:

$$\begin{aligned}
\dot{\theta}_1 &= -2\, d\, J \sin\theta_2 \sin\phi \\
\dot{\theta}_2 &= 2\, d\, J \sin\theta_1 \sin\phi \\
\dot{\phi} &= 2A - 2\, d\, J (\sin\theta_2 \cot\theta_1 - \sin\theta_1 \cot\theta_2) \cos\phi
\end{aligned}$$

where $\phi = \phi_1 - \phi_2$

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Observation of trajectories yields solution for revival time

$$t_{\text{rev}} = \int_{u_1}^{u_2} \mathrm{d}u\, f(u) \quad \text{with} \quad f(u) = \left\{ d^2 J^2 (1-u^2)[1-(2\gamma-u)^2] - (\mathcal{H}_0 - 2Au)^2 \right\}^{-\frac{1}{2}}$$

where $\gamma = 2n - 1$, $\mathcal{H}_0 = -8n(1-n)\,d\,J - 2\gamma A$ and $u_{1/2}$ solutions of $f(u) = 0$

What can we learn from this analytical treatment?

for the revival time:

- dimension rescales J: $J \to dJ$
- scaling: $t_{\text{rev}}(J, A) \equiv t_{\text{rev}}(J/A)/A$
- revival time a solely "energetic" quantity

for the revival amplitude (damping)

- no damping: $n_{k=0} = n - \frac{1}{8dJ}(L-1)(\mathcal{H}_0 + 2A\cos\theta_1)$
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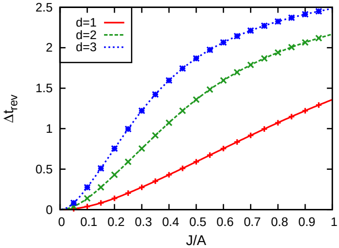

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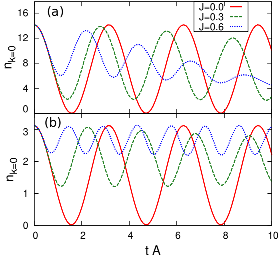
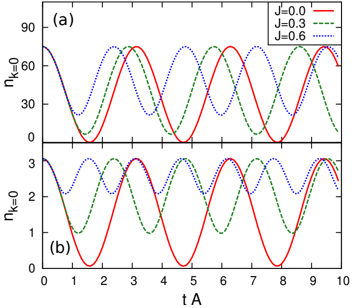
deficencies

- meanfield completely fails to describe revival damping
- artefact for $dJ = 1$: no oscillations

  Sciolla and Biroli, Phys. Rev. Lett. **105** (2010)

need to check validity of the mean-field

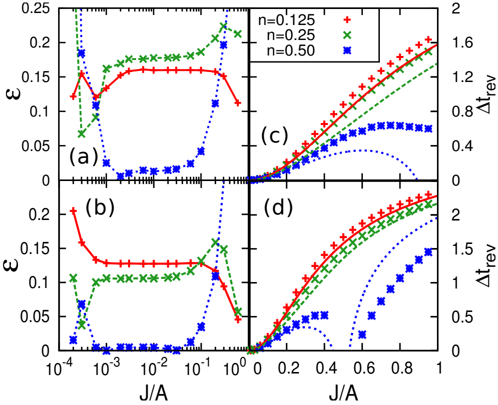




Comparison between exact and mean-field results

error

$$\varepsilon(J) = \frac{\Delta t^{\text{ex}}_{\text{rev}}(J) - \Delta t^{\text{mf}}_{\text{rev}}(J)}{\Delta t^{\text{ex}}_{\text{rev}}(J)}$$



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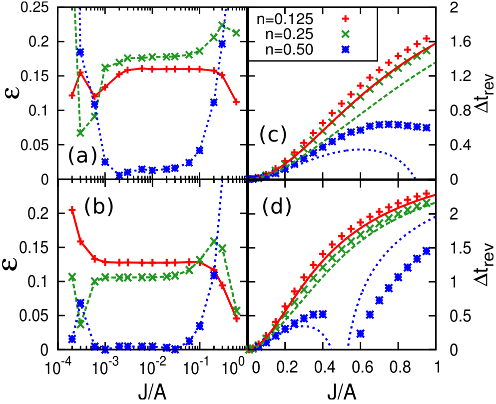

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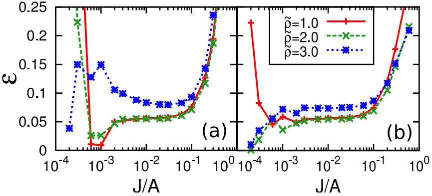
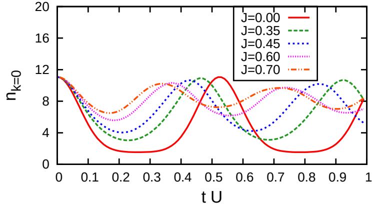
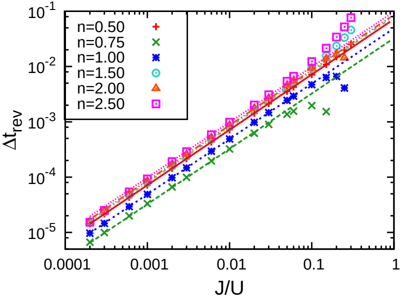


Error for trapped 1D systems: $\sim 5\%$ $\Rightarrow$ Error for 3D systems smaller!

Results for experimentally relevant systems

Results for the Bose-Hubbard model - homogeneous potential





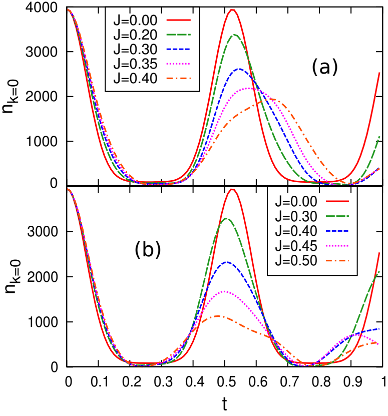
Calculations for exemplary system parameters, in particular an interaction quench of $U_{\text{ini}} = 6 \to U_{\text{fin}} = 12$ and different densities

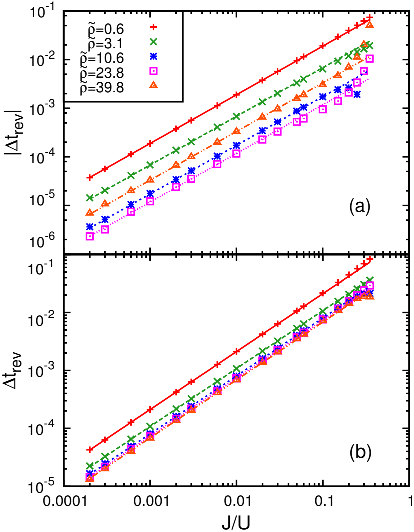
System size

- homogeneous case: one-site problem
- trapped case: calculations for a system with $L = 30 \times 30 \times 30 = 27000$

cut-off for the max. occupancy of a lattice site: $n_{\text{c}} = 7$

Results for the Bose-Hubbard model in a trap





Conclusion

by usage of the analogy: HCBs with super-lattice $\leftrightarrow$ SCBs with interaction
we could extract several reliable results via the application of exact and mean-field approaches:

- very small error of the mean-field for the "right observable", i.e. the revival time
- simple functional form of the relation: $J/U \leftrightarrow t_{\text{rev}}$
- furthermore: definite statements about features such as scaling w.r.t. to system parameters, experimentally meaningful quench scenarios, artefacts of the mean-field

proposed experiment:
for a given value of the interaction constant U, the determination of the hopping constant via a measurement of the revival time is possible by comparison with mean-field calculations

reference: Wolf, Hen, and Rigol, Phys. Rev. A **82**, 043601 (2010)

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